Journal of Combinatorial Theory, Series A 120 (2013) 275–287

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

# Circulant weighing matrices whose order and weight are products of powers of 2 and 3

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ARTICLE INFO

*Article history:*
Received 6 October 2011
Available online 27 August 2012

*Keywords:*
Circulant weighing matrices
Cyclotomic integers
Field descent

ABSTRACT

We classify all circulant weighing matrices whose order and weight are products of powers of 2 and 3. In particular, we show that proper $\mathrm{CW}(v, 36)$'s exist for all $v \equiv 0 \pmod{48}$, all of which are new.



## 1. Introduction

A *circulant weighing matrix of order* $v$ is a square matrix of the form

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_v \\ a_v & a_1 & \cdots & a_{v-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}$$

with $a_i \in \{-1, 0, 1\}$ for all $i$ and $MM^T = nI$ where $n$ is a positive integer and $I$ is the identity matrix. The integer $n$ is called the *weight* of the matrix. A circulant weighing matrix of order $v$ and weight $n$ is denoted by $\mathrm{CW}(v, n)$.

Circulant weighing matrices have been studied intensively, see [2] for a survey and [11] for more background on weighing matrices in general. There are only a few infinite families [3,8,13] and sporadic examples [2,4] of circulant weighing matrices known. Circulant weighing matrices of weight less than or equal to 16 have been classified, see [1,4,5,9,10,19].

In the present paper, we classify all circulant weighing matrices whose order and weight are products of powers of 2 and 3. In principle, this is made possible by the "F-bound" [18] and the results

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0097-3165/$ – see front matter 
http://dx.doi.org/10.1016/j.jcta.2012.08.004

on orthogonal families in [14] which together imply that there is a finite algorithm for this problem. However, we need to employ further tools such as cyclotomic integers of prescribed absolute value and rational idempotents (the latter concept is implicitly used in Section 6) to keep the arguments and computations manageable. We note that all our results are computer free.

The complete classification of circulant weighing matrices whose order and weight are products of powers of 2 and 3 is given in Theorem 6.10 at the end of our paper.

## 2. Preliminaries

Let $C_v$ denote the cyclic group of order $v$. For a divisor $u$ of $v$ we always view $C_u$ as a subgroup of $C_v$.

To study circulant weighing matrices we use the group ring language. The elements of the integral group ring $\mathbb{Z}[C_v]$ have the form

$$X = \sum_{h \in C_v} a_h h$$

with $a_h \in \mathbb{Z}$. The set

$$\{h \in C_v\colon\ a_h \neq 0\}$$

is called the *support* of $X$, and the integers $a_h$ are called the *coefficients* of $X$. We write $|X| = \sum_{h \in C_v} a_h$ and

$$X^{(t)} = \sum_{h \in C_v} a_h h^t$$

for $t \in \mathbb{Z}$. We identify a subset $S$ of $C_v$ with the element $\sum_{h \in S} h$ of $\mathbb{Z}[C_v]$.

A circulant matrix $M$ as defined in Section 1 satisfies $MM^T = nI$ if and only if $DD^{(-1)} = n$ where $D$ is the element of $\mathbb{Z}[C_v]$ defined by $D = \sum_{i=1}^{v} a_i g^i$. Thus a circulant weighing matrix of order $v$ and weight $n$ is equivalent to an element $D$ of $\mathbb{Z}[C_v]$ with coefficients $-1$, 0, 1 only and $DD^{(-1)} = n$. This is the formulation we will use in the rest of our paper. Note that the weight of a circulant weighing matrix must be a square as $|\sum a_i|^2 = n$.

For every multiple $w$ of $v$, any $\mathrm{CW}(v, n)$ can trivially be embedded in $\mathbb{Z}[C_w]$ and thus be viewed as a $\mathrm{CW}(w, n)$. One usually ignores these trivial extensions by concentrating on *proper* circulant weighing matrices, i.e. circulant weighing matrices $D \in \mathbb{Z}[C_v]$ for which there is no $g \in C_v$ and no proper divisor $u$ of $v$ with $Dg \in \mathbb{Z}[C_u]$.

In this paper, we call two circulant weighing matrices $D, E \in \mathbb{Z}[C_v]$ *equivalent* if there are $t, x \in \mathbb{Z}$ with $(t, v) = 1$ and $E = \pm D^{(t)} + x$.

For an abelian group $G$, we denote the group of complex characters of $G$ by $G^*$. The following is a standard result [6, Chapter VI, Lemma 3.5].

**Result 2.1.** Let $G$ be a finite abelian group and $D = \sum_{g \in G} d_g g \in \mathbb{C}[G]$. Then

$$d_g = \frac{1}{|G|} \sum_{\chi \in G^*} \chi\left(Dg^{-1}\right)$$

for all $g \in G$.

The next result is a well-known consequence of Result 2.1.

**Result 2.2.** Suppose $D \in \mathbb{Z}[C_v]$ has coefficients $\pm 1$, 0 only. Then $D$ is a $\mathrm{CW}(v, n)$ if and only if $|\chi(D)|^2 = n$ for all $\chi \in C_v^*$.

We will need the following result on kernels of characters on group rings. See [12, Theorem 2.2] for a proof.

**Result 2.3.** Let $\chi$ be a character of $C_v$ of order $v$. Then the kernel of $\chi$ on $\mathbb{Z}[C_v]$ is

$$\left\{\sum_{i=1}^{r} C_{p_i} X_i\colon\ X_i \in \mathbb{Z}[C_v]\right\}$$

where $p_1, \ldots, p_r$ are the distinct prime divisors of $v$.

For a prime $p$ and a positive integer $t$ let $\nu_p(t)$ be defined by $p^{\nu_p(t)}||t$, i.e. $p^{\nu_p(t)}$ is the highest power of $p$ dividing $t$. By $\mathcal{D}(t)$ we denote the set of prime divisors of $t$. The following definition is necessary for the application of the field descent method [17] which we will do in the next section.

**Definition 2.4.** Let $v$, $n$ be integers greater than 1. For $q \in \mathcal{D}(n)$ let

$$v_q := \begin{cases} \prod_{p\in\mathcal{D}(v)\setminus\{q\}} p & \text{if } v \text{ is odd or } q = 2, \\ 4\prod_{p\in\mathcal{D}(v)\setminus\{2,q\}} p & \text{otherwise.} \end{cases}$$

Set

$$b(2, v, n) = \max_{q\in\mathcal{D}(n)\setminus\{2\}} \left\{\nu_2\left(q^2 - 1\right) + \nu_2\left(\operatorname{ord}_{v_q}(q)\right) - 1\right\} \quad \text{and}$$

$$b(r, v, n) = \max_{q\in\mathcal{D}(n)\setminus\{r\}} \left\{\nu_r\left(q^{r-1} - 1\right) + \nu_r\left(\operatorname{ord}_{v_q}(q)\right)\right\}$$

for primes $r > 2$ with the convention that $b(2, v, n) = 2$ if $\mathcal{D}(n) = \{2\}$ and $b(r, v, n) = 1$ if $\mathcal{D}(n) = \{r\}$. We define

$$F(v, n) := \gcd\left(v, \prod_{p\in\mathcal{D}(v)} p^{b(p,v,n)}\right).$$

The following result was proved in [17].

**Result 2.5.** Assume $X\overline{X} = n$ for $X \in \mathbb{Z}[\zeta_m]$ where $n$ and $m$ are positive integers. Then

$$X\zeta_m^j \in \mathbb{Z}[\zeta_{F(m,n)}]$$

for some $j$.

The following is [18, Theorem 3.2.3]. By $\varphi$ we denote the Euler totient function.

**Result 2.6** *(F-bound).* Let $X \in \mathbb{Z}[\zeta_m]$ be of the form

$$X = \sum_{i=0}^{m-1} a_i \xi_m^i$$

with $0 \leqslant a_i \leqslant C$ for some constant $C$ and assume that $n := X\overline{X}$ is an integer. Then

$$n \leqslant \frac{C^2 F(m, n)^2}{4\varphi(F(m, n))}.$$

**Definition 2.7.** Let $p$ be a prime, let $m$ be a positive integer, and write $m = p^a m'$ with $(p, m') = 1$, $a \geqslant 0$. If there is an integer $j$ with $p^j \equiv -1 \pmod{m'}$, then $p$ is called *self-conjugate* modulo $m$. A composite integer $n$ is called self-conjugate modulo $m$ if every prime divisor of $n$ has this property.

**Result 2.8.** (See Turyn [20].) Assume that $A \in \mathbb{Z}[\zeta_m]$ satisfies

$$A\bar{A} \equiv 0 \mod t^{2b}$$

where $b$, $t$ are positive integers, and $t$ is self-conjugate modulo $m$. Then

$$A \equiv 0 \mod t^b.$$

The following result is due to Ma [15], see also [6, VI, Corollary 13.5] or [16, Corollary 1.2.14].

**Result 2.9** *(Ma).* Let $p$ be a prime and let $G$ be a finite abelian group with a cyclic Sylow $p$-subgroup $S$. If $Y \in \mathbb{Z}[G]$ satisfies

$$\chi(Y) \equiv 0 \mod p^a$$

for all characters $\chi$ of G of order divisible by $|S|$, then there exist $X_1, X_2 \in \mathbb{Z}[G]$ such that

$$Y = p^a X_1 + P X_2,$$

where $P$ is the unique subgroup of order $p$ of $G$.

The next result is [14, Theorem 4.3].

**Result 2.10.** Let $v = w\prod_{i=1}^r p_i^{a_i}$ where the $a_i$'s and $w$ are positive integers and the $p_i$'s are distinct primes coprime to $w$. Let $g$ be a generator of $C_v$. Let $b_i \leqslant a_i$ be positive integers, write $k = \prod_{i=1}^r p_i^{b_i}$. Suppose that $X \in \mathbb{Z}[C_v]$ with the property for every $\tau \in C_v^*$ there is a root of unity $\eta(\tau)$ with

$$\eta(\tau)\tau(X) \in \mathbb{Z}[\zeta_{wk}].$$

Furthermore, assume that $|\tau(X)|^2 \leqslant n$ for all $\tau \in C_v^*$ for some constant $n$. Write $k' = w\prod_{i=1}^r p_i^{c_i}$ where

$$c_i = \begin{cases} \min\{a_i, b_i + \log n/\log p_i\} & \text{if } \log n/\log p_i \text{ is an integer and,} \\ \min\{a_i, \lceil b_i - 1 + \log n/\log p_i\rceil\} & \text{otherwise.} \end{cases}$$

Then $X = \sum_{i=0}^{v/k'-1} X_i g^i$ with $X_i \in \mathbb{Z}[C_{k'}]$, and $X_i X_j = 0$ for all $i \neq j$.

We will need the following result of Kronecker. See [7, Section 2.3, Theorem 2] for a proof.

**Result 2.11.** An algebraic integer all of whose conjugates have absolute value 1 is a root of unity.

## 3. Results

We start with a lemma on cyclotomic integers of prescribed absolute value.

**Lemma 3.1.** *Write $\beta = 1 + \zeta_8 + \zeta_8^3$. Let $v = 2^a \cdot 3^b$ for some nonnegative integers a, b, and let $X \in \mathbb{Z}[\zeta_v]$ with $|X|^2 = 9$. Then there is a root of unity $\eta$ such that*

$$X\eta \in \{3, (\zeta_3 - \zeta_3^2)\beta, (\zeta_3 - \zeta_3^2)\bar{\beta}, \beta^2, \bar{\beta}^2\}. \tag{1}$$

*Furthermore, if $Y \in \mathbb{Z}[\zeta_v]$ with $|Y|^2 = 36$, then $Y = 2Z$ for some $Z \in \mathbb{Z}[\zeta_v]$ with $|Z|^2 = 9$.*

**Proof.** By [18, Theorem 2.2.2] there is a root of unity $\zeta$ such that $X\zeta \in \mathbb{Z}[\zeta_8]$ or $X = (\zeta_3 - \zeta_3^2)Y$ with $Y \in \mathbb{Z}[\zeta_8]$ and $|Y|^2 = 3$.

Case 1: $X\zeta \in \mathbb{Z}[\zeta_8]$. The prime ideal factorization of (3) over $\mathbb{Z}[\zeta_8]$ is $(3) = (\beta)(\bar{\beta})$ (see e.g. [7] for the background in algebraic number theory). Hence $(Z) = (3)$, $(Z) = (\beta^2)$, or $(Z) = (\bar{\beta}^2)$. Now Result 2.11 implies $X\eta \in \{3, \beta^2, \bar{\beta}^2\}$ for some root of unity $\eta$.

Case 2: $X = (\zeta_3 - \zeta_3^2)Y$ with $Y \in \mathbb{Z}[\zeta_8]$ and $|Y|^2 = 3$. Similarly as in Case 1 we conclude $(Y) = (\beta)$ or $(Y) = (\bar{\beta})$ and thus $X\eta \in \{(\zeta_3 - \zeta_3^2)\beta, (\zeta_3 - \zeta_3^2)\bar{\beta}\}$ for some root of unity $\eta$. This proves (1).

The last statement of Lemma 3.1 follows from Result 2.8 since 2 is self-conjugate modulo $3^b$. □

**Lemma 3.2.** *Suppose both $n$ and $v$ are products of powers of 2 and 3. If a $\mathrm{CW}(v,n)$ exists, then $n \leqslant 64$.*

**Proof.** Let $D$ be a $\mathrm{CW}(v,n)$, i.e. $D \in \mathbb{Z}[C_v]$ with coefficients $\pm 1$, 0 and $DD^{(-1)} = n$. We use the F-bound to establish an upper bound on $n$ as follows. Recall that we assume $v$ and $n$ have no prime divisors different from 2 or 3. Let us first assume that $v$ and $n$ are both divisible by 6. Using Definition 2.4, we find $v_2 = 3$ and $v_3 = 4$. Hence

$$b(2, v, n) = \nu_2\big(3^2 - 1\big) + \nu_2\big(\mathrm{ord}_4(3)\big) - 1 = 3 + 1 - 1 = 3$$

and

$$b(3, v, n) = \nu_3\big(2^2 - 1\big) + \nu_3\big(\mathrm{ord}_3(2)\big) = 1.$$

Hence $F(v,n)$ divides 24 by Definition 2.4. It is straightforward to check that $F(v,n)$ also divides 24 if $v$ and $n$ are not both divisible by 6. Hence $F(v,n)$ divides 24 in all cases.

Let $\chi$ be a character of $C_v$ of order $v$. Then $|\chi(D)|^2 = n$ and $\chi(D) = \sum_{i=0}^{v-1} a_i \zeta_v^i$ with $|a_i| \leqslant 1$. Since $\sum_{i=0}^{v-1} \zeta_v^i = 0$, we have $\chi(D) = \sum_{i=0}^{v-1} (a_i + 1)\zeta_v^i$ with $0 \leqslant 1 + a_i \leqslant 2$. Thus the F-bound implies

$$n \leqslant \frac{2^2 \cdot 24^2}{4\varphi(24)} = 72.$$

Since $n$ is a square, we conclude $n \leqslant 64$. □

**Theorem 3.3.** *Suppose both $n$ and $v$ are products of powers of 2 and 3. If a $\mathrm{CW}(v,n)$ exists, then $n \in \{4, 9, 36\}$.*

**Proof.** By Lemma 3.2 it suffices to show $n \neq 16$ and $n \neq 64$. Thus assume $n \in \{16, 64\}$. Since 2 is self-conjugate modulo $v$, we have

$$\chi(D) \equiv 0 \pmod 4 \tag{2}$$

by Result 2.8. If $v$ is odd, then $D \equiv 0 \pmod 4$ by Result 2.1. This is impossible since $D$ has coefficients $\pm 1$, 0 only. Hence $v$ is even. In view of (2), Ma's Lemma implies

$$D = 4X + PY$$

with $X, Y \in \mathbb{Z}[C_v]$ where $P$ is the subgroup of $C_v$ of order 2. But this means that the coefficients of $D$ are constant modulo 4 on each coset of $P$. Since $D$ has coefficients $\pm 1$, 0 only, this shows that, in fact, that the coefficients of $D$ are constant on each coset of $P$. Thus $D = PZ$ with $Z \in \mathbb{Z}[C_v]$. But then $\chi(D) = 0$ for every character $\chi$ of $C_v$ which is nontrivial on $P$. This contradicts $DD^{(-1)} = n$. Hence $n \notin \{16, 64\}$. □

In the following sections, we treat the cases $n = 4, 9, 36$ separately.

## 4. Weight 4

All circulant weighing matrices of weight 4 have been classified in [9]:

**Result 4.1.** Let $D$ be a proper $\mathrm{CW}(v,4)$. Then one of the following occurs.

(i) $v > 2$, $v \equiv 0 \pmod 2$ and $D$ is equivalent to $(1+g) + (1-g)h$ where $g$ is an element of $C_v$ of order 2 and $h \in C_v \setminus \langle g \rangle$.
(ii) $v = 7$ and $D$ is equivalent to $-1 + k^3 + k^5 + k^6$ where $k$ is a generator of $C_7$.

## 5. Weight 9

Let $D$ be a proper $\mathrm{CW}(v,9)$ where $v$ is a product of powers of 2 and 3. By [1, Theorem 3] (see also [19]), we have $v=24$.

**Theorem 5.1.** *Let $\alpha$ and $\delta$ be elements of order 3, respectively 8, in $C_{24}$. Every $\mathrm{CW}(24,9)$ is equivalent to*

$$-1+\left(1-\delta^4\right)\left(\delta+\delta^3\right)+\left(\alpha+\alpha^2\right)\left(1+\delta^4\right).$$

**Proof.** Let $D$ be a $\mathrm{CW}(24,9)$, and let $\chi$ be a character of $C_{24}$ of order 24. By Lemma 3.1 we can assume

$$\chi(D)\in\left\{3,\left(\zeta_3-\zeta_3^2\right)\beta,\left(\zeta_3-\zeta_3^2\right)\bar{\beta},\beta^2,\bar{\beta}^2\right\}$$

where $\beta=1+\zeta_8+\zeta_8^3$. Suppose $\chi(D)\in\{3,(\zeta_3-\zeta_3^2)\beta,(\zeta_3-\zeta_3^2)\bar{\beta}\}$. Then $\chi(D)\equiv 0 \pmod{1-\zeta_3}$, i.e. $\chi(D)=(1-\zeta_3)X$ for some $X\in\mathbb{Z}[\zeta_{24}]$. Let $a$ be the element order 3 of $C_{24}$ with $\chi(a)=\zeta_3$, and choose $A\in\mathbb{Z}[C_{24}]$ with $\chi(A)=X$. By Result 2.3, we have $D=(1-a)A+YC_2+ZC_3$ for some $Y,Z\in\mathbb{Z}[C_{24}]$. Let $\tau=\chi^3$. Then $\tau(C_2)=0$, $\tau(C_3)=3$, and $\tau(a)=1$. Hence $\tau(D)\equiv 0 \pmod 3$. This implies $\psi(D)\equiv 0 \pmod 3$ for all characters $\psi$ of $C_{24}$ of order 8. Note that 3 is self-conjugate modulo 4. Hence by Result 2.8, $\psi(D)\equiv 0 \pmod 3$ for all characters of $C_{24}$ of order dividing 4. In summary, we have shown $\psi(D)\equiv 0 \pmod 3$ for characters of $C_{24}$ of order dividing 8. In view of Result 2.1, this implies $\rho(D)\equiv 0 \pmod 3$ where $\rho: C_{24}\to C_{24}/C_3$ is the natural epimorphism. But since $D$ has coefficients $\pm1$ and 0 only, this implies $D=(1-a)B+TC_3$ for some $B,T\in\mathbb{Z}[C_{24}]$ with coefficients 0, $\pm1$ only, where the elements in the support of $B$ are in distinct cosets of $C_3$, and $T\in\mathbb{Z}[C_8]$. Since $|\gamma(D)|^2=9$ for all $\gamma\in C_{24}^*$, we conclude $|\gamma(T)|=1$ for all $\gamma\in C_{24}^*$ with $\gamma(a)=1$. This implies $T=\pm g$ for some $g\in C_8$ by Result 2.1. As the support of $D$ contains exactly 9 elements and the supports of $(1-a)B$ and $TC_3$ must be disjoint, this implies that the support of $B$ contains exactly 3 elements. As $D=(1-a)B+TC_3$, we have $|\gamma(B)|^2=3$ for all $\gamma\in C_{24}^*$ which are nontrivial on $C_3$. But this is not possible since the support of $B$ contains exactly 3 elements, a contradiction.

Hence we have shown $\chi(D)\in\{\beta^2,\bar{\beta}^2\}$. Replacing $D$ by $D^{(-1)}$, if necessary, we can assume

$$\chi(D)=\beta^2=-1+2\zeta_8+2\zeta_8^3.$$

Furthermore, by replacing $D$ by $-D\delta^4$ if necessary, we can assume $|D|=3$. Moreover, we can choose the element $\delta$ of $C_{24}$ of order 8 such that $\chi(\delta)=\zeta_8$. Result 2.3 shows that

$$D=-1+2\delta+2\delta^3+XC_2+YC_3 \tag{3}$$

with $X,Y\in\mathbb{Z}[C_{24}]$. Let $\tau=\chi^3$. Then

$$\tau(D)=-1+2\zeta_8+2\zeta_8^3+3\tau(Y)\equiv -1+2\zeta_8+2\zeta_8^3 \pmod{1-\zeta_3}.$$

In view of Lemma 3.1, this implies $\tau(D)=-1+2\zeta_8+2\zeta_8^3$ and thus $\tau(Y)=0$. Using Result 2.3 we conclude $YC_3=Y'C_2C_3$ for some $Y'\in\mathbb{Z}[C_{24}]$. Hence we can rewrite (3) as

$$D=-1+2\delta+2\delta^3+ZC_2 \tag{4}$$

for some $Z\in\mathbb{Z}[C_{24}]$. Since $D$ has coefficients $\pm1$ and 0 only, all elements of $\delta C_2\cup\delta^3C_2$ must have coefficient $-1$ in $ZC_2$. Hence we can rewrite (4) as

$$D=-1+\delta+\delta^3-\delta^5-\delta^7+Z'C_2 \tag{5}$$

for some $Z'\in\mathbb{Z}[C_{24}]$ such that $Z'C_2$ and $\delta+\delta^3-\delta^5-\delta^7$ have disjoint supports. Note that $|Z'|=2$ since we assume $|D|=3$. Since the support of $D$ consists only of 9 elements, this implies that the support of $Z'$ consists of exactly 2 elements. Now let $\psi$ be a character of $C_{24}$ of order 3, 6, or 12. Then $\psi(D)=-1+2\psi(Z')$ by (5). Furthermore, by Lemma 3.1, there is a root of unity $\eta(\psi)$ such that

$$\psi(D)=-1+2\psi\left(Z'\right)=3\eta(\psi).$$

Hence $1+\eta(\psi) \equiv 0 \pmod 2$ which implies $\eta(\psi) = \pm 1$. Suppose $\eta(\psi) = 1$. Then $\psi(Z') = 2$. But since the support of $Z'$ only contains 2 elements, this implies that the support of $Z'$ is contained in $C_8$. But then the support of $D$ is contained in $C_8$ which is impossible. Thus $\eta(\psi) = -1$ and $\psi(Z') = -1$ for all characters $\psi$ of $C_{24}$ of order 3, 6, and 12. It is straightforward to check that this implies $Z'C_2 = (\alpha+\alpha^2)C_2$. Substituting this into (5) completes the proof. □

## 6. Weight 36

**Lemma 6.1.** *Assume that $v$ is a product of powers of 2 and 3, and that a $\mathrm{CW}(v, 36)$ exists. Then $v \equiv 0 \pmod 8$.*

**Proof.** Let $D$ be a $\mathrm{CW}(v, 36)$. If $v$ is not divisible by 8, then 3 is self-conjugate modulo $v$. Let $Q$ be the subgroup of $C_v$ of order 3. We have $D = 3X + QY$ with $X, Y \in \mathbb{Z}[C_v]$ by Results 2.8, 2.9. Now the same argument as in the proof of Theorem 3.3 leads to a contradiction. □

We will use the following notation in the next result. Let $S \subset C_v$ and $A = \sum_{h\in C_v} a_h h \in \mathbb{Z}[C_v]$. We write

$$A \cap S := \sum_{h\in A} a_h h.$$

**Lemma 6.2.** *Let $D$ be a $\mathrm{CW}(v, 36)$ where $v = 2^a \cdot 3^b$ and $a, b$ are nonnegative integers. Let $g$ be the element of $C_v$ of order 2, and let $\rho: C_v \to C_v/\langle g\rangle$ be the natural epimorphism.*

*Then $a \geqslant 4$, $b \geqslant 1$, and there is an $h \in C_v$ such that*

$$Dh = (1-g)X + (1+g)Y \tag{6}$$

*with $X, Y \in \mathbb{Z}[C_v]$ such that $\rho(Y)$ is a $\mathrm{CW}(24, 9)$. Furthermore, $X$ has coefficients $\pm 1$, 0 only and its support consists of representatives of distinct cosets of $\langle g\rangle$ in $C_v$.*

**Proof.** Note that 2 is self-conjugate modulo $v$ and thus $\chi(D) \equiv 0 \pmod 2$ by Result 2.8. Hence, if $a = 0$, then $D \equiv 0 \pmod 2$ by Result 2.1, a contradiction. We conclude $a \geqslant 1$.

Ma's Lemma implies

$$D = 2A + (1+g)B$$

with $A, B \in \mathbb{Z}[C_v]$. We can assume that the support of $B$ consists of representatives of distinct cosets of $\langle g\rangle$ in $C_v$, and that all coefficients of $B$ are in $\{-1, 0, 1\}$. This implies that $A$ also has coefficients $-1$, 0, 1 only.

Now fix $k \in C_v$ and suppose that $k$ has coefficient 1 in $B$. Then $A \cap \{k, kg\} \in \{0, -k, -kg, -k-kg\}$, i.e., $D \cap \{k, gk\} \in \{0, (1+g)k, \pm(1-g)k\}$. Similarly, if $k$ has coefficient 0 or $-1$ in $B$, then $D \cap \{k, gk\} \in \{0, -(1+g)k, \pm(1-g)k\}$, too. Now let $Y'$ be the sum of all terms $\pm(1+g)k$ which occur as $D \cap \{k, gk\}$ when $k$ runs through a complete set of coset representatives of $\langle g\rangle$ in $C_v$, and let $X'$ be the sum of all terms $\pm(1-g)k$ which occur. Set $X = X'/(1-g)$ and $Y = Y'/(1+g)$. Then

$$D = (1-g)X + (1+g)Y, \tag{7}$$

$X$ and $Y$ have coefficients $\pm 1$, 0 only, and the support of $X$ consists of representatives of distinct cosets of $\langle g\rangle$ in $C_v$.

It remains to show that we can find $w \in C_v$ such that $\rho(Yw)$ is a $\mathrm{CW}(24, 9)$. Note that $\rho(Y)$ has coefficients $\pm 1$, 0 only since the support of $Y$ consists of representatives of distinct cosets of $\langle g\rangle$. Furthermore, $\chi(Y) = \chi(D)/2$ for all characters of $C_v$ which are trivial on $\langle g\rangle$ by (7). Hence $\rho(Y)$ is a $\mathrm{CW}(v/2, 9)$ by Result 2.2 (we are identifying $C_v/\langle g\rangle$ with $C_{v/2}$ here). By [1, Theorem 3] there is a $u \in C_{v/2}$ such that $\rho(Y)u$ is a $\mathrm{CW}(24, 9)$. This concludes the proof. □

**Lemma 6.3.** *We use the notation of Lemma 6.2. Write $u = 2^8 \cdot 3^4$, and let $v' = \operatorname{lcm}(u, v)$. There are a positive integer $k$ and $Z_1, \dots, Z_k \in C_u$, $a_1, \dots, a_k \in C_{v'}$ such that*

$$(1-g)X = \sum_{i=1}^{k} Z_i a_i \tag{8}$$

*and $Z_i Z_j = 0$ for $i \neq j$. Furthermore, the supports of the elements $Z_i a_i$, $i = 1, \dots, k$, are pairwise disjoint.*

**Proof.** Note that we can view $D$ as an element of $\mathbb{Z}[C_{v'}]$. Since $|\chi(D)|^2 = 36$ for all characters $\chi$ of $C_{v'}$, we have $|\chi((1-g)X)|^2 = 36$ for all characters $\chi$ of $C_{v'}$ with $\chi(g) = -1$ and $|\chi((1-g)X)|^2 = 0$ for all characters $\chi$ of $C_{v'}$ with $\chi(g) = 1$. Furthermore, by Lemma 3.1, we have $\chi(D)\eta \in \mathbb{Z}[\zeta_{24}]$ for some root of unity $\eta$ (depending on $\chi$) for all characters $\chi$ of $C_{v'}$.

In summary, we have shown $|\chi((1-g)X)|^2 \leqslant 36$ and $\chi((1-g)X)\eta \in \mathbb{Z}[\zeta_{24}]$ for some root of unity $\eta$ for all characters $\chi$ of $C_{v'}$. Hence we can apply Result 2.10 with $w = 1$, $p_1 = 2$, $p_2 = 3$, $b_1 = 3$, $b_2 = 1$, $k = 24$, $n = 36$, and find $c_1 = 8$, $c_2 = 4$. Hence

$$(1-g)X = \sum_{i=0}^{v'/u-1} X_i \alpha^i$$

with $X_i \in \mathbb{Z}[C_u]$ and $X_i X_j = 0$ for $i \neq j$ where $\alpha$ is a generator of $C_v$. This implies (8). □

Fix $i$ and let $Z_i$ be from (8). Note that $Z_i$ has coefficients $\pm 1$, 0 only because $(1-g)X$ has coefficients $\pm 1$, 0 only. Recall that $|\chi((1-g)X)|^2 = 36$ for all characters $\chi$ of $C_{v'}$ with $\chi(g) = -1$ and $|\chi((1-g)X)|^2 = 0$ for all characters $\chi$ of $C_{v'}$ with $\chi(g) = 1$. Hence, in summary,

$$\begin{aligned}
&Z_i \in \mathbb{Z}[C_u] \text{ and } Z_i \text{ has coefficients } \pm 1, 0 \text{ only,} \\
&\chi(Z_i) = 0 \text{ for all characters } \chi \text{ of } C_u \text{ with } \chi(g) = 1 \text{ and} \\
&\big|\chi(Z_i)\big| \in \{0, 36\} \text{ for all characters } \chi \text{ of } C_u \text{ with } \chi(g) = -1.
\end{aligned} \tag{9}$$

**Lemma 6.4.** *Let $u = 2^8 \cdot 3^4$ and assume that $Z_i \in \mathbb{Z}[C_u]$ satisfies (9). Let $\chi$ be any character of $C_u$. Then there is a root of unity $\eta$ such that*

$$\chi(Z_i)\eta \in \big\{0, 6, 2\big(\zeta_3 - \zeta_3^2\big)\beta, 2\big(\zeta_3 - \zeta_3^2\big)\bar{\beta}\big\}$$

*where $\beta = 1 + \zeta_8 + \zeta_8^3$.*

**Proof.** Assume the contrary. Then, by Lemmas 3.1 and (9), we have

$$\chi(Z_i)\eta \in \big\{2\beta^2, 2\bar{\beta}^2\big\} \tag{10}$$

for some root of unity $\eta$. By replacing $Z_i$ by $Z_i^{(a)} g$ for some $a \in \{-1, 1\}$, $g \in C_v$, if necessary, we can assume

$$\chi(Z_i) = 2\beta^2. \tag{11}$$

Note that the order of $\chi$ must be divisible by $2^8$ by (9) and (11). Let $\tau$ be a character of $C_u$ of order $3^b$, $0 \leqslant b \leqslant 4$. Write $Z_i = \sum_{g \in C_u} a_g g$ with $a_g \in \mathbb{Z}$. Then

$$\chi\tau(Z_i) - \chi(Z_i) = \sum_{g \in C_u} a_g \chi(g)\big(\tau(g) - 1\big).$$

Note that $\tau(g) - 1 \equiv 0 \pmod{1 - \zeta_{81}}$ for all $g$ because $\tau(g)$ is an 81st root of unity. Hence

$$\chi\tau(Z_i) \equiv \chi(Z_i) \pmod{1 - \zeta_{81}}. \tag{12}$$

We claim that $2\beta^2 \not\equiv 0 \pmod{1-\zeta_{81}}$. Assume that contrary, i.e., $-2+4\zeta_8+4\zeta_8^3=(1-\zeta_{81})T$ with $T \in \mathbb{Z}[\zeta_{8\cdot 81}]$. Write $T=\sum_{i=0}^{3}\zeta_8^i T_i$ with $T_i \in \mathbb{Z}[\zeta_{81}]$. Then

$$-2+4\zeta_8+4\zeta_8^3=\sum_{i=0}^{3}\zeta_8^i T_i(1-\zeta_{81}).$$

Since $\{1,\zeta_8,\zeta_8^2,\zeta_8^3\}$ is linearly independent over $\mathbb{Q}(\zeta_{81})$, we get $2 \equiv 0 \pmod{1-\zeta_{81}}$, a contradiction. Thus

$$2\beta^2 \not\equiv 0 \pmod{1-\zeta_{81}}. \tag{13}$$

Similarly, we see that

$$2\left(\beta^2-\eta\bar{\beta}^2\right) \not\equiv 0 \pmod{1-\zeta_{81}} \tag{14}$$

for all $u$th roots of unity $\eta$. Recall that $|\tau\chi(Z_i)| \in \{0,36\}$ by (9) and thus

$$\tau\chi(Z_i)\eta \in \left\{0,6,2\left(\zeta_3-\zeta_3^2\right)\beta,2\left(\zeta_3-\zeta_3^2\right)\bar{\beta},2\beta^2,2\bar{\beta}^2\right\} \tag{15}$$

for some root of unity $\eta$ by Lemma 3.1. However, if $\tau\chi(Z_i) \neq 2\beta^2$, then $\tau\chi(Z_i)-\chi(Z_i) \not\equiv 0 \pmod{1-\zeta_{81}}$ by (11), (13), (14), and (15). This contradicts (12). Hence

$$\tau\chi(Z_i)\eta = 2\beta^2 \tag{16}$$

for some root of unity $\eta$.

Recall that the order of $\chi$ is divisible by $2^8$ and that the above arguments works for every character $\tau$ of $C_u$ of order dividing 81. Hence, in summary, we have shown

$$\psi\tau(Z_i)\eta = 2\beta^2 \tag{17}$$

for some root of unity $\eta$ (depending on $\psi$ and $\tau$) for all characters $\psi$ of order $2^8$ and all characters $\tau$ of order dividing 81. Let $\tau_0$ be a character of order 81, and let $\psi$ be a character of order $2^8$. W.l.o.g. assume $\psi\tau_0(Z_i)=2\beta^2$. By Result 2.3, the kernel of $\psi\tau_0$ on $\mathbb{Z}[C_u]$ is

$$\left\{AC_2+BC_3\colon\ A,B \in \mathbb{Z}[C_u]\right\}.$$

Hence

$$Z_i = 2\left(-1+2x^2+2x^4\right)+AC_2+BC_3 \tag{18}$$

with $A,B \in \mathbb{Z}[C_u]$ where $x$ is the element of $C_u$ of order 8 with $\psi(x)=\zeta_8$. Applying $\psi\tau^3$ to (18), we find $\psi\tau^3(Z_i)=2\beta^2+3\psi\tau_1(B)$. Recall that $\psi\tau^3(Z_i)=2\bar{\eta}\beta^2$ for some root of unity $\eta$ by (17). Hence $2\beta^2(1-\eta) \equiv 0 \pmod 3$. This implies $\eta=1$. Hence $\psi\tau^3(Z_i)=2\beta^2$. Let $\rho: C_u \to C_u/C_3$ be the natural epimorphism. Using the same argument again, with $Z_i$ replaced by $\rho(Z_i)$, $\tau$ replaced by $\tau^3$, and $\tau^3$ replaced by $\tau^9$, we find $\psi\tau^9(Z_i)=2\beta^2$. Continuing this, we see that $\psi\tau^{3^a}(Z_i)=2\beta^2$ for $a=0,\dots,4$. Hence, applying Galois automorphisms of $\mathbb{Q}(\zeta_{81})$ to these identities, we find

$$\psi\tau^j(Z_i)=2\beta^2 \tag{19}$$

for $j=0,\dots,80$. Now write $Z_i=\sum_{k=0}^{80}A_k t^k$ with $A_k \in \mathbb{Z}[C_{2^8}]$ where $t$ is an element of $C_u$ of order 81. Then

$$\sum_{j=0}^{80}\psi\tau^j(Z_i)=\sum_{j=0}^{80}\sum_{k=0}^{80}\psi\tau^j(A_k)\psi\tau^j\left(t^k\right)=\sum_{k=0}^{80}\psi(A_k)\sum_{j=0}^{80}\tau^j\left(t^k\right)=81\psi(A_0).$$

Combining this with (19), we find $\psi(A_0) = 2\beta^2$. Note that $A_0$ has coefficients $\pm 1$, 0 only because $Z_i$ has coefficients $\pm 1$, 0 only. Hence $\psi(A_0) = \sum_{i=0}^{255} a_i \zeta_{256}^i$ with $|a_i| \leqslant 1$. Note that $1, \zeta_{256}, \ldots, \zeta_{256}^{127}$ are linearly independent over $\mathbb{Q}$ and

$$2\beta^2 = -2 + 4\zeta_8 + 4\zeta_8^3 = \psi(A_0) = \sum_{i=0}^{127} (a_i - a_{i+128})\zeta_{256}^i.$$

This is a contradiction because $|a_i - a_{i+128}| \leqslant 2 < 4$ for all $i$. □

**Lemma 6.5.** *Let $u = 2^8 \cdot 3^4$ and assume that $Z_i \in \mathbb{Z}[C_u]$ satisfies (9). Let $\chi$ be a character of $C_u$ of order $2^8 t$ where $t$ divides 27, and let $\tau$ be a character of $C_u$ of order $2^8 \cdot 3^4$. Then the following hold.*

(a) *There is a root of unity $\eta$ such that $\chi(Z_i)\eta \in \{0, 6\}$.*
(b) *There is a root of unity $\eta$ such that $\tau(Z_i)\eta \in \{0, 2(\zeta_3 - \zeta_3^2)\beta, 2(\zeta_3 - \zeta_3^2)\bar{\beta}\}$.*

**Proof.** Let $\sigma$ be a character of $C_u$ of order $2^8 t$ such that $\chi = \sigma^3$. By Lemma 6.4 we have $\sigma(Z_i) \equiv 0 \pmod{1 - \zeta_3}$. Let $K = \{h \in C_u\colon\ \sigma(h) = 1\}$ and let $\rho : C_u \to C_u/K$ be the canonical epimorphism. Note that $\sigma$ and thus $\chi$ can be viewed as a character of $C_u/K$, too. By Result 2.3, the kernel of $\sigma$ on $\mathbb{Z}[C_u/K]$ is

$$\{XU_2 + YU_3\colon\ X, Y \in \mathbb{Z}[C_u/K]\}$$

where $U_2$ and $U_3$ are subgroups of order 2, respectively 3, of $C_u/K$. Hence $\rho(Z_i) = (1 - a)A + XU_2 + YU_3$ with $A, X, Y \in \mathbb{Z}[C_u/K]$ and where $a$ is an element of $C_u/K$ with $\sigma(a) = \zeta_3$. Note that $\chi(a) = \sigma(a)^3 = 1$, $\chi(U_2) = 0$, and $\chi(U_3) = 3$. Hence

$$\chi(Z_i) = \chi\big(\rho(Z_i)\big) = 3\chi(Y) \equiv 0 \pmod 3.$$

In view of Lemma 6.4, this implies (a).

Now assume that (b) does not hold. Then by part (a), (9), and Lemma 6.4, we have $\psi(Z_i) \equiv 0 \pmod 6$ for all characters $\psi$ of $C_u$. Thus $Z_i = 3X + C_3 Y$ with $X, Y \in \mathbb{Z}[C_u]$ by Ma's Lemma. Since $Z_i$ has coefficients $\pm 1$, 0 only, this implies that $Z_i$ is a multiple of $C_3$. But this implies $\tau(Z_i) = 0$ contradicting our assumption. □

**Lemma 6.6.** *Let $u = 2^8 \cdot 3^4$ and assume that $Z_i \in \mathbb{Z}[C_u]$ satisfies (9). Let $\alpha$ and $\delta$ be elements of $C_u$ of order 3, respectively 8. Write*

$$A = \big(1 + \delta + \delta^3\big)\big(\alpha - \alpha^2\big), \qquad B = \big(1 - \delta - \delta^3\big)\big(\alpha - \alpha^2\big).$$

*There are $c, d \in C_u$ and $x, y \in \{-1, 0, 1\}$ such that*

$$Z_i = \big(1 - \delta^4\big)(cxA + dyC_3) \quad \text{or} \quad Z_i = \big(1 - \delta^4\big)(xcB + dyC_3). \tag{20}$$

**Proof.** For $j = 0, \ldots, 4$, let $\chi_j$ be a character of $C_u$ of order $2^8 \cdot 3^j$. By Lemma 6.5 there are $h_j \in C_u$ and $\epsilon_j \in \{-1, 0, 1\}$ such that

$$\chi_j(Z_i) = 6\epsilon_j \chi_j(h_j), \quad j = 0, \ldots, 3. \tag{21}$$

Let $h_4$ be any element of $\mathbb{Z}[C_u]$ with

$$\chi_4(h_4) = \chi_4(Z_i)/2. \tag{22}$$

We claim that

$$\begin{aligned} 27Z_i = {} & \big(1 - \delta^4\big)\big[\epsilon_0 h_0 C_{81} + \epsilon_1 h_1 (3C_{27} - C_{81}) + \epsilon_2 h_2 (9C_9 - 3C_{27}) \\ & + \epsilon_3 h_3 (27C_3 - 9C_9) + h_4 (27 - 9C_3)\big]. \end{aligned} \tag{23}$$

By Result 2.1, to verify (23), we need to show that the character values of both sides of (23) are the same for all characters of $C_u$. But this follows from (9), (21), and (22). Thus (23) holds.

Considering (23) modulo 3 we find $(1-\delta^4)(\epsilon_0 h_0 C_{81} - \epsilon_1 h_1 C_{81}) \equiv 0 \pmod 3$. This implies $\epsilon_0 h_0 C_{81} = \epsilon_1 h_1 C_{81}$. Similarly, we deduce $\epsilon_1 h_1 C_{27} = \epsilon_2 h_2 C_{27}$ and $\epsilon_2 h_2 C_9 = \epsilon_3 h_3 C_9$. Hence we have

$$27 Z_i = \left(1-\delta^4\right)\left(27\epsilon_3 h_3 C_3 + h_4(27 - 9C_3)\right). \tag{24}$$

If $\chi_4(Z_i) = 0$, then we can choose $h_4 = 0$. Then $Z_i = (1-\delta^4)\epsilon_3 h_3 C_3$ and thus (20) holds.

Now assume $\chi_4(Z_i) \neq 0$. By Lemma 6.5(b) there is a root of unity $\eta$ such that $\chi_4(Z_i)\eta \in \{2(\zeta_3 - \zeta_3^2)\beta, 2(\zeta_3 - \zeta_3^2)\bar{\beta}\}$ where $\beta = 1 + \zeta_8 + \zeta_8^3$. Hence we can choose

$$h_4 = \pm c\left(\alpha - \alpha^2\right)\left(1 + \delta + \delta^3\right) \quad \text{or} \quad h_4 = \pm c\left(\alpha - \alpha^2\right)\left(1 - \delta - \delta^3\right) \tag{25}$$

for some $c \in C_u$. Note $h_4 C_3 = 0$. Thus substituting (25) into (24) gives $Z_i = (1-\delta^4)(\epsilon_3 h_3 C_3 \pm c(\alpha - \alpha^2)(1+\delta+\delta^3))$ or $Z_i = (1-\delta^4)(\epsilon_3 h_3 C_3 \pm c(\alpha - \alpha^2)(1-\delta-\delta^3))$. Thus (20) holds in all cases. □

The following two theorems completely classify circulant weighing matrices $\mathrm{CW}(v, 36)$ where $v$ is a product of a power of 2 and a power of 3.

**Theorem 6.7.** *Let $D$ be a $\mathrm{CW}(v, 36)$ where $v$ is a product of a power of 2 and a power of 3. Let $\alpha$ and $\gamma$ be elements of $C_v$ of order 3, respectively 16. Write*

$$A_1 = \left(1 + \gamma^2 + \gamma^6\right)\left(\alpha - \alpha^2\right),$$
$$A_2 = \left(1 - \gamma^2 - \gamma^6\right)\left(\alpha - \alpha^2\right),$$
$$B = -1 + \left(1 - \gamma^4\right)\left(\gamma + \gamma^3\right) + \left(\alpha + \alpha^2\right)\left(1 + \gamma^4\right).$$

*Then, up to equivalence,*

$$D = \left(1 + \gamma^8\right)B + \left(1 - \gamma^8\right)(cA_i + dC_3) \tag{26}$$

*with $i \in \{1, 2\}$, $c, d \in C_v$. Furthermore, the supports of $B$, $(1-\gamma^8)cA_i$, and $(1-\gamma^8)dC_3$ are pairwise disjoint.*

**Proof.** Using the notation of Lemma 6.2, we can assume

$$D = \left(1 - \gamma^8\right)X + \left(1 + \gamma^8\right)Y \tag{27}$$

with $X, Y \in \mathbb{Z}[C_v]$ such that $\rho(Y)$ is a $\mathrm{CW}(24, 9)$. Furthermore, in view of Theorem 5.1, we can assume $Y = B$. So it only remains to show that $(1-\gamma^8)X$ can be written in the form

$$\left(1 - \gamma^8\right)X = \left(1 - \gamma^8\right)(cA_i + dC_3). \tag{28}$$

Write $u = 2^8 \cdot 3^4$ and $v' = \mathrm{lcm}(u, v)$. Using the notation of Lemma 6.3, we have

$$\left(1 - \gamma^8\right)X = \sum_{i=1}^{k} Z_i a_i \tag{29}$$

with $Z_1, \ldots, Z_k \in C_u$, $a_1, \ldots, a_k \in C_{v'}$. Furthermore, we can assume $Z_i \neq 0$ for all $i$ and that the supports of the elements $Z_i a_i$, $i = 1, \ldots, k$, are pairwise disjoint. Note that, by (27), the support of $(1-\gamma^8)X$ consists of exactly 18 elements since the support of $D$, respectively $Y$, consists of exactly 36 respectively 9, elements. Recall that

$$Z_i = \left(1 - \gamma^8\right)(cxA + dyC_3) \quad \text{or} \quad Z_i = \left(1 - \gamma^8\right)(xcB + dyC_3)$$

by Lemma 6.6 (where $c$, $d$, $x$, $y$ depend on $i$).

Using the notation of Lemma 6.6, we divide the $Z_i$ into types as follows:

Type 1: $x \neq 0,\ y = 0$.
Type 2: $x = 0,\ y \neq 0$.
Type 3: $x \neq 0,\ y \neq 0$.

Suppose $Z_i$ is of Type 3. Note that there cannot be any overlap in the supports of the terms comprising $Z_i$ since otherwise $Z_i$ would have a coefficient $\pm 2$. Hence the support of $Z_i$ consists of exactly 18 elements. This implies $k = 1$, and thus (28) holds.

Now assume there is no $Z_i$ of Type 3. Since the support of $(1-\gamma^8)X$ consists of exactly 18 elements, one of the following must occur.

Case 1: There are three $Z_i$'s of Type 2 and no $Z_i$ of Type 1. But then $\chi(D) = 0$ for all characters of $C_v$ with $\chi(\gamma^8) = -1$ and $\chi(\alpha) = \zeta_3$ contradicting Result 2.2. Thus Case 1 cannot occur.

Case 2: There is exactly one $Z_i$ of Type 1 and exactly one $Z_i$ of Type 2. Then (28) holds.

We have shown that (28) holds in all cases which concludes the proof. □

**Theorem 6.8.** *Write*

$$M_1 = \{\gamma, \gamma^3, \gamma^5, \gamma^7\}, \qquad M_2 = \{\gamma^2, \gamma^6, \gamma^{10}, \gamma^{14}\}, \qquad M_3 = \{\gamma, \gamma^3, \gamma^4, \gamma^5, \gamma^7\}.$$

*If the supports of $B$, $(1-\gamma^8)cA_i$, and $(1-\gamma^8)dC_3$ in (26) are pairwise disjoint, then $D$ is a* CW$(v, 36)$. *This occurs if and only if one of the following holds.*

(i) $c \notin C_{48}$ *and* $d \notin C_{48} \cup C_{48}c$.
(ii) $c \notin C_{48}$ *and* $d \in M_2C_3$.
(iii) $c \notin C_{48}$ *and* $d \in cM_3C_3 \cup c\gamma^8 M_3C_3$.
(iv) $c \in M_1 \cup M_1\gamma^8$ *and* $d \notin C_{48}$.
(v) $c \in M_1 \cup M_1\gamma^8$ *and* $d \in M_2C_3$.

**Proof.** If the supports of $B$, $(1-\gamma^8)cA_i$, and $(1-\gamma^8)dC_3$ in (26) are pairwise disjoint, then $D$ has coefficients $\pm 1$, 0 only. Straightforward checking shows that $|\chi(D)|^2 = 36$ for all characters $\chi$ of $C_v$. Hence $D$ is a CW$(v, 36)$. The necessary and sufficient condition for the disjointness of supports also follows by straightforward checking. □

**Corollary 6.9.** *There exist proper* CW$(v, 36)$ *for all* $v \equiv 0 \pmod{48}$.

**Proof.** Condition (i) of Theorem 6.8 shows that there are proper CW(48, 36)'s. Condition (iv) guarantees the existence of proper CW$(v, 36)$ for all $v > 48$ divisible by 48. □

We conclude our paper by a theorem summarizing our results.

**Theorem 6.10.** *Let $D$ be a proper* CW$(v, n)$ *where both $v$ and $n$ are products of powers of* 2 *and* 3. *Then* $n \in \{4, 9, 36\}$.

(a) *If $n = 4$, then one of the following holds.*

(i) $v > 2$, $v \equiv 0 \pmod 2$ *and $D$ is equivalent to* $(1+g) + (1-g)h$ *where $g$ is an element of $C_v$ of order* 2 *and* $h \in C_v \setminus \langle g \rangle$.
(ii) $v = 7$ *and $D$ is equivalent to* $-1 + k^3 + k^5 + k^6$ *where $k$ is a generator of* $C_7$.

(b) *If $n = 9$, then $v = 24$ and $D$ is equivalent to*

$$-1 + (1-\delta^4)(\delta + \delta^3) + (\alpha + \alpha^2)(1 + \delta^4)$$

*where $\alpha$ and $\delta$ are elements of order* 3, *respectively* 8, *in* $C_{24}$.

(c) *If* $n = 36$, *then* $D$ *is equivalent to*

$$D = \left(1+\gamma^8\right)B + \left(1-\gamma^8\right)(cA_i + dC_3)$$

*where* $i \in \{1,2\}$, $\alpha$ *and* $\gamma$ *are elements of* $C_v$ *of order* 3, *respectively* 16,

$$A_1 = \left(1+\gamma^2+\gamma^6\right)\left(\alpha-\alpha^2\right),$$
$$A_2 = \left(1-\gamma^2-\gamma^6\right)\left(\alpha-\alpha^2\right),$$
$$B = -1 + \left(1-\gamma^4\right)\left(\gamma+\gamma^3\right) + \left(\alpha+\alpha^2\right)\left(1+\gamma^4\right),$$

*and* $c, d \in C_v$, *such that one of the following conditions is satisfied.*

(i) $c \notin C_{48}$ *and* $d \notin C_{48} \cup C_{48}c$.
(ii) $c \notin C_{48}$ *and* $d \in M_2C_3$.
(iii) $c \notin C_{48}$ *and* $d \in cM_3C_3 \cup c\gamma^8M_3C_3$.
(iv) $c \in M_1 \cup M_1\gamma^8$ *and* $d \notin C_{48}$.
(v) $c \in M_1 \cup M_1\gamma^8$ *and* $d \in M_2C_3$.

*Here*

$$M_1 = \left\{\gamma, \gamma^3, \gamma^5, \gamma^7\right\}, \qquad M_2 = \left\{\gamma^2, \gamma^6, \gamma^{10}, \gamma^{14}\right\}, \qquad M_3 = \left\{\gamma, \gamma^3, \gamma^4, \gamma^5, \gamma^7\right\}.$$

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